The Hales-Jewett Theorem

The Hales-Jewett theorem is one of the most important results in Ramsey Theory. It easily implies many other results, and is a useful ingredient in many proofs. To state it, we need some notation.

The n-dimensional t-cube $[t]^n$ is defined as

$$[t]^n = \{(x_1, \ldots, x_n)\} : 1 \leq x_i \leq t\},$$

where of course the x_i are all integers. A subset L of $[t]^n$ is a *combinatorial line* if there exists a nonempty set $I \subset [n]$ and integers a_i for each $i \notin I$ such that

$$L = \{(x_1, \ldots, x_n) \in [t]^n : x_i = a_i \text{ for } i \notin I, \text{ and } x_i = x_j \text{ for } i, j \in I\}.$$

Combinatorial lines are just ordinary lines in the cube, with the additional restriction that, as one moves along the line, all the *active* coordinates (those in I) increase from 1 to t together, while the *fixed* coordinates (those not in I) remain constant. For instance, there are $2t + 1$ combinatorial lines in $[t]^2$: t horizontal lines, t vertical lines, and one (not two) diagonals. From now on, "line" will always mean "combinatorial line". Here is the theorem.

Theorem 1. *For all positive integers r and t, there exists a least integer $n = HJ(r, t)$ such that any r-coloring of $[t]^n$ contains a monochromatic line.*

Proof. As in van der Waerden's theorem, the key idea is color-focusing. Given a line L, write L^- and L^+ for its first and last points (in the obvious ordering). Lines $L_1, \ldots, L_s$ are *focused* at f if $L_i^+ = f$ for all i, and they are *color-focused* at f if all the truncated lines $L_i \setminus \{L_i^+\}$ are monochromatic of different colors. Note that these definitions exactly match those in the proof of van der Waerden's theorem.

A line in $[t]^n$ is specified by its first (or last) point, together with its "direction", i.e., its active coordinate set I. We will exploit this fact to cut down on the amount of notation we need.

Now to the proof itself. We use induction on t. The case $t = 1$ is trivial. Suppose we know that $HJ(m, t-1)$ is finite *for all* m. Our aim is to show that, for a fixed r, $HJ(r,t)$ is also finite.

To do this, we will show, for each $s \leq r$, the existence of a number $N = FHJ(r,s,t)$ such that any r-coloring of $[t]^N$ contains either

- A combinatorial line, or
- s color-focused lines.

The case $s = r$ will imply the theorem, since the focus of r color-focused lines must receive one of the r colors, extending a truncated monochromatic line to a full one: this is the point of color-focusing.

Turning to the assertion, we again use induction on s. The case $s = 1$ is trivial: just take $FHJ(r,1,t) = HJ(r,t-1)$. Assume that we know that $n = FHJ(r,s-1,t)$ is finite. I claim that

$$FHJ(r,s,t) \leq N = n + HJ(r^{t^n}, t-1) =: n + n'.$$

Here is why: suppose we are given an r-coloring χ of $[t]^N$, where $N = n + n'$. Consider this first as a r^{t^n}-coloring χ' of $[t]^{n'}$, by associating each point $b \in [t]^{n'}$ with the entire χ-colored cube $\{(a,b) : a \in [t]^n\}$. By definition of n' (this is the induction on t), there is a line L in $[t]^{n'}$, with active coordinate set I, such that the truncated line $L \setminus \{L^+\}$ is monochromatic under χ'. What this means in terms of the original coloring χ is that, for all $a \in [t]^n$, and all $b, b' \in L \setminus \{L^+\}$, we have

$$\chi((a,b)) = \chi((a,b')) =: \chi''(a).$$

Now we examine the coloring χ'' of $[t]^n$. By hypothesis (this is the induction on s), we can find $s-1$ color-focused lines $L_1, \ldots, L_{s-1}$ in $[t]^n$, with active coordinate sets $I_1, \ldots, I_{s-1}$ and focus f. All we need to do now is put the pieces together. For $1 \leq i \leq s-1$, define L_i' to be the line in $[t]^N$ with first point (L_i^-, L^-) and active coordinate set $I \cup I_i$, and define L_s' to be the line in $[t]^N$ with first point (f, L^-) and active coordinate set I. Unless $[t]^N$ contains a monochromatic line, the lines L_i', for $1 \leq i \leq s$, form a set of s color-focused lines, with focus (f, L^+). $\square$